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DISCRETE-TIME FOURIER GENERALIZED CONVOLUTION INEQUALITY AND TOEPLITZ PLUS HANKEL EQUATION

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Abstract:
In this paper, we study the Young type inequality and the generalized convolution transform for the discrete-time Fourier sine generalized convolution. Solution in closed form for some clases of the Toeplitz plus Hankel equation related to the discrete-time Fourier sine generalized convolution are considered.

Keywords: *Fourier cosine Series, Fourier sine Series, Discrete Convolution, Discrete Young's Inequality, Discrete Toeplitz Plus Hankel Equation.*

1. Introduction

The discrete-time Fourier transform is a transformation that maps discrete-time signal *x(n)* into a complex-valued function of the real variable, namely [1, 2, 3].

$$X(\omega) := F_{DT}\{x(n)\}(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-i\omega n} \in \mathrm{C},\ \omega \in \mathrm{R}.$$

The discrete-time Fourier convolution of *x(n)* and *y(n)* is a sequence, denotes by $(x \underset{F}{*} y)$ and be defined as follows [1, 2]

$$(x \underset{F}{*} y)(n) = \sum_{m=-\infty}^{\infty} x(m)y(n-m),\ \ \mathrm{n} \in \mathrm{Z}. \tag{1.1}$$

However, particular case of discrete-time Fourier transform is a discrete-time Fourier cosine and discrete-time Fourier sine transforms have not been studied.

Recently, we studied discrete-time Fourier cosine transform on $\mathbb{N}_0$ (see [4])

$$X_c(\omega) := F_{cDT}\{x(n)\}(\omega) = \frac{x_o}{2} + \sum_{n=1}^{\infty} x(n)\cos(n\omega),\ \omega \in [0, \pi]. \tag{1.2}$$

equipped with a norm in $l_p(\mathbb{N}_0), 1 \le p \le \infty$,

$$\|x\|_p := \left(\frac{|x(0)|^p}{2} + \sum_{n=1}^{\infty}|x(n)|^p\right)^{\frac{1}{p}} < \infty,\ \ 1 \le p < \infty \tag{1.3}$$

Constructed the commutative convolution has the form (see [4])

$$(x \underset{F_{cDT}}{*} y)(n) = \sum_{m=1}^{\infty} x(m)[y(n+m) + y(n-m)] + x(0)y(n),\ \mathrm{n} \in \mathbb{N}_0. \tag{1.4}$$

which has the following property

$$F_{cDT}\{(x \underset{F_{cDT}}{*} y)(n)\}(\omega) = 2F_{cDT}\{x(n)\}(\omega)\cdot F_{cDT}\{y(n)\}(\omega)\ \forall\omega \in [0, \pi] \tag{1.5}$$

Here, we will establish Young type inequality for the convolution on $\mathbb{N}_0$ with the sharp constant in two important cases $p = q = 1$ and $p = q = 2$.

The Toeplitz plus Hankel integral equation is of the form (see [5, 6])

$$f(x) + \int_0^{\infty} [k_1(x+y) + k_2(x-y)]x(m) = g(n),\ \ \forall x \in \mathbb{R}. \tag{1.6}$$

here g, k_1, k_2 are given and f is a unknown function.

When in discrete form, the equation (1.6) has the form

$$x(n) + \sum_{m=0}^{\infty}[k_1(n+m) + k_2(n-m)]x(m) = g(n),\ n \in \mathbb{N}_0. \tag{1.7}$$

with k_1, k_2, g are given and x is a unknown sequence.

For other examples of discrete-time Toeplitz-Hankel equations that can be solved in closed form see [2]. A special case of equation (1.7) with the Toeplitz kernel $k_2(n) = k(|n|)$, and the Hankel kernel $k_1(n) = k(n)$ has been studied in [4]

$$x(n) + \sum_{m=1}^{\infty}[k(n+m) + k(n-m)]x(m) + x(0)k(n) = g(n),\ n \in \mathbb{N}_0. \tag{1.8}$$

Under certain conditions, the equation (1.8) has the unique solution in $l_1(\mathbb{N}_0)$ (see [4]).

2. Discrete-time Fourier sine transform and inequalities

The discrete-time Fourier sine transform of a sequence $x := \{x(n)\}_{n \ge 1}$ is defined by

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$$X_s(\omega) \equiv F_{sDT}\{x(n)\}(\omega) := \sum_{n=1}^{\infty} x(n)\sin(n\omega),$$
$$\omega \in [0,\pi] \quad (2.1)$$

with a norm in $l_p^0(\mathbb{N}_0), 1 \le p < \infty$ its subspace of (1.3) when x(0) = 0.

Obviously, if $x \in l_p^0(\mathbb{N}_0)$, then $X_s \in L_\infty(0,\pi)$, and $\|X_s\|_\infty \le \|x\|_1$. And if $x \in l_2^0(\mathbb{N}_0)$, then $X_s \in L_2(0,\pi)$, and the Parseval formula for the discrete-time Fourier sine transform yields

$$\|x\|_2^2 = \frac{2}{\pi}\|X_s\|_2^2. \quad (2.2)$$

Definition 1. The generalized convolution $x * y$ of sequences x and y for the discrete-time Fourier sine and Fourier cosine transforms is defined by

$$(x*y)(n) = \sum_{m=1}^{\infty} x(m)[y(|n-m|) - y(n+m)],\ n \in \mathbb{N}_0, \quad (2.3)$$

if series converges for any $n \ge 0$.

Theorem 1. If $x \in l_2^0(\mathbb{N}_0)$ and $y \in l_2(\mathbb{N}_0)$. Then the discrete convolution (2.3) belongs to the space $l_\infty^0(\mathbb{N}_0)$, and moreover,

$$\|x*y\|_\infty \le 2\|x\|_2\|y\|_2, \lim_{n\to\infty}(x*y)(n) = 0. \quad (2.4)$$

The following Parseval formula holds

$$(x*y)(n) = \frac{4}{\pi}\int_0^{\pi} X_s(\omega)Y_c(\omega)\sin(n\omega)d\omega,\ n \ge 0. \quad (2.5)$$

Theorem 2. (*A discrete Young's type theorem*). Let *p, q, r* > 1, satisfy the condition $x \in l_p(\mathbb{N}_0)$, $y \in l_q(\mathbb{N}_0)$, $h \in l_r(\mathbb{N}_0)$, $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 2$ and $x(0) = h(0) = 0$, then

$$\left|\sum_{n=1}^{\infty}(x*y)(n)\cdot h(n)\right| \le \|x\|_{l_p(\mathbb{N}_0)}\cdot\|y\|_{l_q(\mathbb{N}_0)}\cdot\|h\|_{l_r(\mathbb{N}_0)}.$$

Corollary 1 (*A discrete Young's type inequality*). Let *p, q, r* > 1, satisfy the condition $\frac{1}{p}+\frac{1}{q} = 1+\frac{1}{r}$. Let $x \in l_p^0(\mathbb{N}_0)$, $y \in l_p(\mathbb{N}_0)$, then $(x*y) \in l_r^0(\mathbb{N}_0)$, and morever $\|x*y\|_r \le \|x\|_p\|y\|_q$.

Theorem 3. Assume that $x \in l_1^0(\mathbb{N}_0)$, $y \in l_1(\mathbb{N}_0)$ and $x(0) = 0$. Then $(x*y) \in l_1^0(\mathbb{N}_0)$, and factorization equality holds

$$F_{sDT}\{(x*y)(n)\}(\omega) = 2F_{sDT}\{x(n)\}(\omega)\cdot F_{cDT}\{y(n)\}(\omega),$$
$$\forall\omega \in [0,\pi]. \quad (2.8)$$

Moreover,

$$\|x*y\|_1 \le 2\|x\|_1\|y\|_1. \quad (2.9)$$

The equality holds if both x and y are nonnegative (nonpositive) sequences.

3. A discrete Toeplitz plus Hankel equation

We consider the Toeplitz plus Hankel equation

$$x(|n|) + \sum_{m=1}^{\infty} x(m)[k_1(n+m) + k_2(n-m)] = g(n),\ n \in \mathbb{Z}, \quad (3.1)$$

in case the kernel sequences k_1, k_2 are arbitrary and right-hand side satisfies a certain condition. Namely, we obtain the following theorem.

Theorem 4. Given that $g_1, g_2, k_1, k_2 \in l_1(\mathbb{N}_0)$, $g_1(0) = 0$, $g = g_1 + g_2$, and satisfy the conditions

$$1 + 2F_{cDT}\{(k_1+k_2)(n)\}(\omega) \ne 0,\ \forall\omega \in [0,\pi], \quad (3.2)$$

and

$$g_1(n) = \left(\left(\left(g_2 \underset{F_{cDT}}{*} l\right) - g_2\right) * (k_1 - k_2)\right)(n), \quad (3.3)$$

here $l \in l_1(\mathbb{N}_0)$ is defined by

$$F_{cDT}\{l(n)\}(\omega) = \frac{F_{cDT}\{(k_1+k_2)(n)\}(\omega)}{1 + 2F_{cDT}\{(k_1+k_2)(n)\}(\omega)}.$$

Then the integral equation (3.1) has the unique solution in $l \in l_1(\mathbb{N}_0)$, which is of the form:

$$x(n) = g_2(n) - \left(g_2 \underset{F_{cDT}}{*} l\right)(n).$$

Proof. Extend g_1 to the whole $\mathbb{Z}$ as an odd sequence, *x*, g_2 as even sequences, and extend *g* to the whole $\mathbb{Z}$ by the rule $g(n) = g_1(n) + g_2(n)$. Equation (3.1) can be rewritten in the form

$$x(|n|) + \frac{1}{2}\sum_{m=1}^{\infty}\{[k_1(|n+m|) + k_2(|n+m|)$$
$$+ k_1(|n-m|) + k_2(|n-m|)] + [k_1(n+m)$$
$$- k_2(|n+m|) - k_1(|n-m|)$$
$$+ k_2(|n-m|)]\}x(m) = g(n),\ n \in \mathbb{Z}. \quad (3.4)$$

Applying the discrete-time Fourier transform to both sides of equation (3.4) and note that $F_{DT}\{h(n)\}(\omega) = 2F_{cDT}\{h(n)\}(\omega)$, $\omega \in [-\pi,\pi]$, if h is even sequence, $F_{DT}\{h(n)\}(\omega) = 2F_{sDT}\{h(n)\}(\omega)$, $\omega \in [-\pi,\pi]$ if h is odd sequence. Using equations (1.4), (1.5) and Theorem 3, we obtain

$$2F_{cDT}\{x(n)\}(\omega) + 4F_{cDT}\{x(n)\}(\omega)F_{cDT}\{k_1(n)+k_2(n)\}(\omega)$$
$$+ i4F_{sDT}\{x(n)\}(\omega)F_{cDT}\{k_1(n)-k_2(n)\}(\omega)$$
$$= F_{DT}\{g(n)\}(\omega),$$
$$\omega \in [-\pi,\pi]. \quad (3.5)$$

Recall that $g(n) = g_1(n) + g_2(n)$, where g_1, g_2 respectively are even and odd components of *g*. Therefore *x* is a solution of equation (3.5) if and only if the both of following conditions are satisfied

$$F_{cDT}\{x(n)\}(\omega) + 2F_{cDT}\{x(n)\}(\omega)F_{cDT}\{k_1(n)+k_2(n)\}(\omega)$$
$$= F_{cDT}\{g_2(n)\}(\omega), \quad (3.6)$$

and

$$2F_{sDT}\{x(n)\}(\omega)F_{cDT}\{k_1(n)-k_2(n)\}(\omega)$$
$$=F_{sDT}\{g_1(n)\}(\omega). \quad (3.7)$$

Equation (3.7) can be rewritten in the form

$$F_{cDT}\{x(n)\}(\omega)$$
$$=2F_{cDT}\{g_2(n)\}(\omega)\left(1-\frac{2F_{cDT}\{k_1(n)+k_2(n)\}(\omega)}{1+2F_{cDT}\{k_1(n)+k_2(n)\}(\omega)}\right). \quad (3.8)$$

In virtue of the Wiener-Levy's type Theorem for Fourier cosine series (see [4]), by the given condition (3.2), there exists a unique sequence $l\in l_1(\mathbb{N}_0)$ such that

$$F_{cDT}\{l(n)\}(\omega)=\frac{F_{cDT}\{k_1(n)+k_2(n)\}(\omega)}{1+2F_{cDT}\{k_1(n)+k_2(n)\}(\omega)}.$$

Therefore, from (3.8) we have

$$F_{cDT}\{x(n)\}(\omega)=2F_{cDT}\{g_2(n)\}(\omega)[1-2F_{cDT}\{l(n)\}(\omega)].$$

Derive

$$x(n)=g_2(n)-(g_2 \underset{F_{cDT}}{*} \mathrm{l})(\mathrm{n}), \quad \mathrm{n}\in\mathbb{N}. \quad (3.9)$$

Substitute (3.9) into (3.8) we have

$$2\Big(F_{sDT}\{g_2(n)\}(\omega)-F_{sDT}\{(g_2 \underset{F_{cDT}}{*} l)(n)\}(\omega)\Big)F_{cDT}.$$
$$.\{k_1(n)-k_2(n)\}(\omega)$$
$$=-F_{sDT}\{g_1(n)\}(\omega),$$

or

$$F_{sDT}\{(g_2*(k_1-k_2))(n)\}(\omega)$$
$$-F_{sDT}\Big\{\Big((g_2 \underset{F_{cDT}}{*} l)*(k_1-k_2)\Big)(n)\Big\}(\omega)$$
$$=-F_{sDT}\{g_1(n)\}(\omega).$$

Therefore,

$$g_1(n)=\Big(\big((g_2 \underset{F_{cDT}}{*} \mathrm{l})-g_2\big)*(k_1-k_2)\Big)(\mathrm{n}), \quad \mathrm{n}\in\mathbb{N}.$$

From (3.5), (3.6), (3.7) and (3.9) we obtain solution of equation (3.1) in $l\in l_1(\mathbb{N}_0)$ in this form

$$x(n)=g_2(n)-(g_2 \underset{F_{cDT}}{*} \mathrm{l})(\mathrm{n}), \quad \mathrm{n}\in\mathbb{N}.$$

The proof is completed.

Disclosure statement

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BẤT ĐẲNG THỨC TÍCH CHẬP SUY RỘNG FOURIER THỜI GIAN RỜI RẠC VÀ PHƯƠNG TRÌNH TOEPLITZ CỘNG HANKEL

Tóm tắt:

Trong bài báo này, chúng tôi nghiên cứu bất đẳng thức kiểu Young và biến đổi tích chập suy rộng cho tích chập Fourier sine thời gian rời rạc. Ứng dụng giải một lớp phương trình Toeplitz cộng Hankel liên quan tới tích chập suy rộng Fourier tời gian rời rạc.

Keywords: *Chuỗi Fourier cosine, Chuỗi Fourier sine, Tích chập rời rạc, Bất đẳng thức Young's rời rạc, Phương trình Toeplitz cộng Hankel rời rạc.*